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Spline collocation methods for linear multi-term fractional differential equations☆

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ABSTRACT

Some regularity properties of the solution of linear multi-term fractional differential equations are derived. Based on these properties, the numerical solution of such equations by piecewise polynomial collocation methods is discussed. The results obtained in this paper extend the results of Pedas and Tamme (2011) [15] where we have assumed that in the fractional differential equation the order of the highest derivative of the unknown function is an integer. In the present paper, we study the attainable order of convergence of spline collocation methods for solving general linear fractional differential equations using Caputo form of the fractional derivatives and show how the convergence rate depends on the choice of the grid and collocation points. Theoretical results are verified by some numerical examples.

1. Introduction

In this paper, we study the convergence behavior of a collocation method for the numerical solution of multi-term fractional differential equations of the form

$$D_*^{\alpha_p} y(t) + \sum_{i=0}^{p-1} a_i(t) D_*^{\alpha_i} y(t) = f(t), \quad 0 \le t \le b, \tag{1.1}$$

equipped with the initial conditions

$$y^{(i)}(0) = \gamma_i, \quad i = 0, \dots, n-1. \tag{1.2}$$

In (1.1)

$$0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_p, \quad n-1 < \alpha_p \le n, \; n, \; p \in \mathbb{N} = \{1, 2, \dots\}, \tag{1.3}$$

a_i ($i = 0, \dots, p-1$) and f are some given continuous functions from $[0, b]$ into $\mathbb{R} = (-\infty, \infty)$, $D_*^0 = I$ is the identity operator and $D_*^\alpha y$ is used to represent the Caputo-type fractional derivative of order $\alpha > 0$ defined by (see, e.g., [1])

$$D_*^\alpha y(t) = D^\alpha (y - Q_{k-1}[y])(t), \quad t > 0,$$

where the integer $k \ge 1$ is such that $k - 1 < \alpha \le k$,

$$Q_{k-1}[y](s) = \sum_{i=0}^{k-1} \frac{y^{(i)}(0)}{i!} s^i$$

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and $D^\alpha y$ is the Riemann–Liouville fractional derivative of order α:

$$D^\alpha y(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}(J^{k-\alpha}y)(t), \quad k-1 < \alpha \le k,\ k \in \mathbb{N},\ t > 0.$$

Here $J^0 = I$ and J^α, the Riemann–Liouville integral operator, is defined for $\alpha > 0$ by the formula

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s, \quad t > 0, \tag{1.4}$$

where $\Gamma(\alpha)$ is the Euler gamma function. If $\alpha = k \in \mathbb{N}$ then $D^k y = D_*^k y = y^{(k)}$ where $y^{(k)}$ is the usual k-th order derivative of y.

It is well known (see, e.g., [2,3]) that J^α, $\alpha > 0$, is linear, bounded and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$, and for any $y \in L^\infty(0, b)$

$$\begin{aligned}
&(J^\alpha y)^{(k)} \in C[0,b], \qquad (J^\alpha y)^{(k)}(0) = 0, \quad \alpha > 0,\ k = 0, \dots, \lceil \alpha \rceil - 1,\\
&J^\alpha J^\beta y = J^{\alpha+\beta} y, \quad \alpha > 0,\ \beta > 0,\\
&D^\beta J^\alpha y = D_*^\beta J^\alpha y = J^{\alpha-\beta} y, \quad 0 < \beta \le \alpha,
\end{aligned} \tag{1.5}$$

where $\lceil \alpha \rceil$ is the smallest integer not less than α.

The problems of fractional differential equations arise in various areas of science and engineering. In particular, multi-term fractional differential equations have been used to model various types of visco-elastic damping (see [4]). In the last few decades theory and numerical analysis of fractional differential equations have received an increasing attention (see, for example, [1,3–5] and references cited in these books). We refer also to the recent papers [6–10] where fractional linear multi-step methods and predictor–corrector methods of Adams type for solving Cauchy problems of fractional differential equations are discussed. Fractional linear multi-step methods for weakly singular Volterra integral equations are studied in [11].

Our aim is to establish an effective way to approximate a solution of Eq. (1.1) using a spline collocation method. These methods have shown to be efficient to solve integral and integro-differential equations (see, e.g., [2,12–14]) but have received very little attention for solving fractional differential equations [15–20]. From these papers only in [15] also the convergence rate of proposed algorithms has been investigated. More precisely, in [15] the attainable order of convergence of a spline collocation method for solving (1.1) with (1.2) under the assumption that α_p is an integer has been studied. The purpose of the present paper is to extend the discussion to the case of a non-integer value of α_p and to refine some results of [15]. Note also that our analysis will be carried out in a situation where the derivatives of the functions $f(t)$ and $a_i(t)$ $(i = 0, \dots, p-1)$ of Eq. (1.1) may be unbounded at $t = 0$.

The remainder of the present paper is arranged as follows. In Section 2 we prove Theorem 2.1 which gives the estimates for higher order derivatives of the exact solution of Eq. (1.1) satisfying (1.2). These estimates will play a key role in the convergence analysis of proposed algorithms in Section 4. In Section 3 the description of a piecewise polynomial collocation method is given. We use an integral equation reformulation of the problem and special non-uniform grids reflecting the possible singular behavior of the exact solution. In Section 4 we prove the convergence of our method, derive global convergence estimates and analyze a (global) superconvergence effect for a special choice of collocation points. The main results of the paper are formulated in Theorems 4.1 and 4.2. Finally, in Section 5 the obtained theoretical results are verified by some numerical experiments.

2. Smoothness of the solution

In order to characterize the behavior of higher order derivatives of a solution of Eq. (1.1), we introduce a weighted space of smooth functions $C^{q,\nu}(0, b]$ (cf., e.g., [2]). For given $q \in \mathbb{N}$ and $-\infty < \nu < 1$, by $C^{q,\nu}(0, b]$ we denote the set of continuous functions $y : [0, b] \to \mathbb{R}$ which are q times continuously differentiable in $(0, b]$ and such that for all $t \in (0, b]$ and $i = 1, \dots, q$ the following estimates hold:

$$|y^{(i)}(t)| \le c \begin{cases} 1 & \text{if } i < 1-\nu,\\ 1 + |\log t| & \text{if } i = 1-\nu,\\ t^{1-\nu-i} & \text{if } i > 1-\nu, \end{cases}$$

where $c = c(y)$ is a positive constant. Equipped with the norm

$$\|y\|_{q,\nu} = \max_{0\le t\le b}|y(t)| + \sum_{i=1}^{q} \sup_{0<t\le b}(w_{i+\nu-1}(t)|y^{(i)}(t)|),$$

$C^{q,\nu}(0, b]$ is a Banach space. Here

$$w_\lambda(t) = \begin{cases} 1 & \text{for } \lambda < 0,\\ (1 + |\log t|)^{-1} & \text{for } \lambda = 0,\\ t^\lambda & \text{for } \lambda > 0. \end{cases}$$

Clearly,

$$C^q[0,b] \subset C^{q,\nu}(0,b] \subset C^{m,\mu}(0,b] \subset C[0,b], \quad q \geq m \geq 1,\ \nu \leq \mu < 1.$$

Note that a function of the form $y(t) = g_1(t)\, t^\mu + g_2(t)$ is included in $C^{q,\nu}(0,b]$ if $\mu \geq 1-\nu > 0$ and $g_j \in C^{q,\nu}(0,b],\ j = 1, 2$.

In what follows we use the following reformulation of the problem (1.1), (1.2). Let y be a solution of Eq. (1.1) satisfying (1.2) and set

$$Q(t) = \sum_{j=0}^{n-1} \frac{\gamma_j}{j!} t^j. \tag{2.1}$$

Then the function $\tilde{y} = y - Q$ satisfies the homogeneous initial conditions

$$\tilde{y}^{(i)}(0) = 0, \quad i = 0, \dots, n-1.$$

Moreover, we have

$$D_*^{\alpha_p} \tilde{y}(t) + \sum_{i=0}^{p-1} a_i(t) D_*^{\alpha_i} \tilde{y}(t) = \tilde{f}(t), \quad 0 < t \leq b,$$

where

$$\tilde{f}(t) = f(t) - \sum_{i=0}^{p-1} a_i(t) D_*^{\alpha_i} Q(t). \tag{2.2}$$

It is well known (see, e.g., [1]) that for $t > 0$

$$D_*^{\alpha_i} Q(t) = \begin{cases} \displaystyle\sum_{j=\lceil \alpha_i \rceil}^{n-1} \frac{\gamma_j}{\Gamma(1+j-\alpha_i)} t^{j-\alpha_i} & \text{if } 0 \leq \alpha_i \leq n-1, \\ 0 & \text{if } \alpha_i > n-1. \end{cases} \tag{2.3}$$

Assume that $z = D_*^{\alpha_p} \tilde{y} \in C[0,b]$. Then $\tilde{y} = J^{\alpha_p} z$ (see [1,3]). Using (1.5) we see that z is a solution of Eq.

$$z = Tz + \tilde{f} \tag{2.4}$$

where

$$T = -\sum_{i=0}^{p-1} A_i J^{\alpha_p - \alpha_i}, \quad A_i v = a_i v. \tag{2.5}$$

It follows from (2.5) and (1.4) that (2.4) is a Volterra-type integral equation of the second kind. This equation is equivalent to the problem (1.1), (1.2) in following sense: if $y \in C^{n-1}[0,b]$ with $D_*^{\alpha_p} y \in C[0,b]$ is a solution of Eq. (1.1) satisfying (1.2), then $z = D_*^{\alpha_p} y = D_*^{\alpha_p} \tilde{y}$ is a solution to (2.4); conversely, if $z \in C[0,b]$ is a solution to (2.4) then $y = J^{\alpha_p} z + Q$ is a solution of the problem (1.1), (1.2).

Theorem 2.1. *Let (1.3) be true and assume that $a_i \in C^{q,\mu}(0,b]$ $(i = 0, \dots, p-1)$ and $f \in C^{q,\mu}(0,b]$, where $q \in \mathbb{N}$ and $-\infty < \mu < 1$. Then problem (1.1), (1.2) possesses a unique solution $y \in C^{n-1}[0,b]$ such that $D_*^{\alpha_p} y \in C[0,b]$. For this solution there holds $D_*^{\alpha_p} y \in C^{q,\nu}(0,b]$ where*

$$\nu = \begin{cases} \max\{\mu, \nu_1, \nu_2\} & \text{if } Q \neq 0, \\ \max\{\mu, \nu_1\} & \text{if } Q = 0, \end{cases} \tag{2.6}$$

with

$$\nu_1 = \max\{1 - \alpha_p + \alpha_i : \alpha_p - \alpha_i \notin \mathbb{N},\ i = 0, \dots, p-1\},$$

$$\nu_2 = \max\{1 - \lceil \alpha_i \rceil + \alpha_i : \alpha_i < n-1, \alpha_i \notin \mathbb{N}_0,\ i = 0, \dots, p-1\}.$$

Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and Q is defined by (2.1). If for all indices $i = 0, \dots, p-1$ we have $\alpha_p - \alpha_i \in \mathbb{N}$ then $\nu = \max\{\mu, \nu_2\}$ for $Q \neq 0$ and $\nu = \mu$ for $Q = 0$. Analogously, if we have $\alpha_i \in \mathbb{N}_0$ for all indices $i = 0, \dots, p-1$ such that $\alpha_i < n-1$ then $\nu = \max\{\mu, \nu_1\}$ or $\nu = \mu$ if, in addition, $\alpha_p - \alpha_i \in \mathbb{N}$, $i = 0, \dots, p-1$.

Proof. From (2.3) it follows that $D_*^{\alpha_i} Q \in C^q[0,b]$ if $\alpha_i \in \mathbb{N}_0$, $D_*^{\alpha_i} Q = 0$ if $\alpha_i > n-1$ and $D_*^{\alpha_i} Q \in C^{q,\nu_2}(0,b]$ if $\alpha_i \notin \mathbb{N}_0,\ 0 < \alpha_i < n-1$. Therefore in (2.4) $\tilde{f} \in C^{q,\tilde{\nu}}(0,b]$ with $\tilde{\nu} = \max\{\mu, \nu_2\}$. If $Q = 0$ then $\tilde{f} = f \in C^{q,\mu}(0,b]$. In a similar way as in the proof of Theorem 2.1 in [15] we can show that T defined by (2.5) is linear and compact as an operator from $C[0,b]$ into $C[0,b]$ and also from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$ and Eq. (2.4) has a unique solution $z \in C^{q,\nu}(0,b] \subset C[0,b]$. Consequently, the problem (1.1), (1.2) possesses a unique solution $y = J^{\alpha_p} z + Q \in C^{n-1}[0,b]$ such that $D_*^{\alpha_p} y = z \in C^{q,\nu}(0,b] \subset C[0,b]$. □

Remark 2.1. If in Eq. (1.1) $a_i \in C^q[0,b]$ $(i = 0, \dots, p-1)$ and $f \in C^q[0,b]$ where $q \in \mathbb{N}$ then in Theorem 2.1 we may set μ to be equal to any number which is less than 1.

3. Spline collocation method

For $N \in \mathbb{N}$, let $\Pi_N := \{t_0, \ldots, t_N\}$ be a partition (a graded grid) of the interval $[0, b]$ with the grid points

$$t_j = b\left(\frac{j}{N}\right)^r, \quad j = 0, 1, \ldots, N, \tag{3.1}$$

where the grading exponent $r \in \mathbb{R}$, $r \geq 1$. If $r = 1$, then the grid points (3.1) are distributed uniformly; for $r > 1$ the points (3.1) are more densely clustered near the left endpoint of the interval $[0, b]$.

For given integer $m \geq 0$ by $S_m^{(-1)}(\Pi_N)$ is denoted the standard space of piecewise polynomial functions on $[0, b]$:

$$S_m^{(-1)}(\Pi_N) = \{v : v|_{(t_{j-1},t_j)} \in \pi_m,\ j = 1, \ldots, N\}.$$

Here $v|_{(t_{j-1},t_j)}$ is the restriction of $v : [0, b] \to \mathbb{R}$ onto the subinterval (t_{j-1}, t_j) and π_m denotes the set of polynomials of degree not exceeding m. Note that the elements of $S_m^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \ldots, t_{N-1}$ of the grid Π_N.

In every interval $[t_{j-1}, t_j]$, $j = 1, \ldots, N$, we define $m \in \mathbb{N}$ collocation points $t_{j1}, \ldots, t_{jm}$ by formula

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \ldots, m,\ j = 1, \ldots, N, \tag{3.2}$$

where $\eta_1 \ldots, \eta_m$ are some fixed (collocation) parameters which do not depend on j and N and satisfy

$$0 \leq \eta_1 < \eta_2 < \cdots < \eta_m \leq 1. \tag{3.3}$$

We look for an approximate solution y_N of the Cauchy problem (1.1), (1.2) in the form

$$y_N = J^{\alpha_p} z_N + Q \tag{3.4}$$

where Q is defined by (2.1) and $z_N \in S_{m-1}^{(-1)}(\Pi_N)(m, N \in \mathbb{N})$ is determined by the following collocation conditions:

$$z_N(t_{jk}) = (Tz_N)(t_{jk}) + \tilde{f}(t_{jk}), \quad k = 1, \ldots, m,\ j = 1, \ldots, N. \tag{3.5}$$

Here $\tilde{f}$, T and t_{jk} are defined by (2.2), (2.5) and (3.2), respectively. If $\eta_1 = 0$, then by $z_N(t_{j1})$ we denote the right limit $\lim_{t\to t_{j-1}, t>t_{j-1}} z_N(t)$. If $\eta_m = 1$, then $z_N(t_{jm})$ denotes the left limit $\lim_{t\to t_j, t<t_j} z_N(t)$. Conditions (3.5) have an operator equation representation:

$$z_N = \mathcal{P}_N T z_N + \mathcal{P}_N \tilde{f} \tag{3.6}$$

with an interpolation operator $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0, T] \to S_{m-1}^{(-1)}(\Pi_N)$ defined for any $v \in C[0, b]$ by the following conditions:

$$\mathcal{P}_N v \in S_{m-1}^{(-1)}(\Pi_N), \quad (\mathcal{P}_N v)(t_{jk}) = v(t_{jk}),\ k = 1, \ldots, m,\ j = 1, \ldots, N. \tag{3.7}$$

Collocation conditions (3.5) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$. We refer to [15] for a convenient choice of it.

Having determined the approximation z_N for $z = D_*^{\alpha_p} y$, we can determine the approximation $y_N = J^{\alpha_p} z_N + Q$ for the solution y of Eq. (1.1) satisfying (1.2) and the approximations $y_N^{(i)} = J^{\alpha_p - i} z_N + Q^{(i)}$ for the derivatives $y^{(i)} = J^{\alpha_p - i} z + Q^{(i)}$, $i = 1, \ldots, n-1$.

4. Convergence analysis

First we prove the convergence of our method and study the attainable order of convergence for arbitrary collocation parameters $\eta_1, \ldots, \eta_m$ satisfying (3.3). In what follows by c and c_1 we denote positive constants that are independent of $N \in \mathbb{N}$ and may have different values in different occurrences. For given Banach spaces X and Y, by $\mathcal{L}(X, Y)$ we denote the Banach space of bounded linear operators $A : X \to Y$ with the norm $\|A\|_{\mathcal{L}(X,Y)} = \sup\{\|Az\|_Y : z \in X, \|z\|_X \leq 1\}$.

Theorem 4.1. *Let (1.3) be true and assume that $a_i \in C[0, b]$ $(i = 0 \ldots, p-1)$ and $f \in C[0, b]$. Let $m \in \mathbb{N}$ and assume that the collocation points (3.2) with grid points (3.1) and arbitrary parameters $\eta_1, \ldots, \eta_m$ satisfying (3.3) are used.*

Then problem (1.1), (1.2) has a unique solution $y \in C^{n-1}[0, b]$ such that $D_*^{\alpha_p} y \in C[0, b]$ and there exists an integer N_0 such that for all $N \geq N_0$ Eq. (3.6) possesses a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and

$$\max_{0\leq i\leq n-1} \|y^{(i)} - y_N^{(i)}\|_\infty \to 0 \quad \text{as } N \to \infty \tag{4.1}$$

where y_N is defined by (3.4). If, in addition, the assumptions of Theorem 2.1 with $q = m$ are fulfilled, then for all $N \geq N_0$ and $r \geq 1$ the following error estimate holds:

$$\max_{0\leq i\leq n-1} \|y^{(i)} - y_N^{(i)}\|_\infty \leq c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r < \dfrac{m}{1-\nu}, \\ N^{-m}(1+\log N) & \text{for } r = \dfrac{m}{1-\nu} = 1, \\ N^{-m} & \text{for } r = \dfrac{m}{1-\nu} > 1 \\ & \text{or } r > \dfrac{m}{1-\nu}. \end{cases} \tag{4.2}$$

Here ν is defined by (2.6) and

$$\|v\|_\infty = \sup_{0<t<b} |v(t)|, \quad v \in L^\infty(0, b).$$

Proof. Since T defined by (2.5) is linear and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$ and Eq. $z = Tz$ has in $C[0, b]$ only the trivial solution $z = 0$, Eq. (2.4) has a unique solution $z \in C[0, b]$. Consequently, the problem (1.1) and (1.2) possesses a unique solution $y = J^{\alpha_p} z + Q \in C^{n-1}[0, b]$ such that $D_*^{\alpha_p} y = z \in C[0, b]$. Using a standard argument (see, e.g., [2,15]) we obtain that there exists an integer N_0 such that for $N \geq N_0$ the operators $(I - \mathcal{P}_N T)$ are invertible in $L^\infty(0, b)$,

$$\|(I - \mathcal{P}_N T)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c, \quad N \geq N_0,$$

and Eq. (3.6) possesses a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$. From (2.4) and (3.6) we get that

$$(I - \mathcal{P}_N T)(z - z_N) = z - \mathcal{P}_N z, \quad N \geq N_0,$$

and consequently

$$\|z - z_N\|_\infty \leq c\, \|z - \mathcal{P}_N z\|_\infty, \quad N \geq N_0.$$

Since $y^{(i)} = J^{\alpha_p - i} z + Q^{(i)}$, $y_N^{(i)} = J^{\alpha_p - i} z_N + Q^{(i)}$,

$$\|y^{(i)} - y_N^{(i)}\|_\infty \leq c\|z - z_N\|_\infty \leq c_1\|z - \mathcal{P}_N z\|_\infty, \quad N \geq N_0,\ i = 0 \dots, n-1,$$

and $\|z - \mathcal{P}_N z\|_\infty \to 0$ for every $z \in C[0, b]$ as $N \to \infty$ (see [2]), we have justified the convergence (4.1). If the assumptions of Theorem 2.1 with $q = m$ are fulfilled then $z \in C^{m,\nu}(0, b]$ where ν is defined by (2.6), and with the help of Lemma 3.1 of [2] we get the estimate (4.2). $\square$

In order to study the superconvergence properties of our method by a special choice of collocation parameters we need Lemma 4.3 which generalizes Lemmas 4.1 and 4.2 of [15]. In the proof of Lemma 4.3 we use the following elementary Lemmas 4.1 and 4.2 which proofs can be found in [12] and in [21,22], respectively.

Lemma 4.1. *Let $z \in C^{m,\nu}(0, b]$, $m \in \mathbb{N}$ and $-\infty < \nu < 1$. Then*

$$\sup_{t_{j-1}<s<t_j} |(z - \mathcal{P}_N z)(s)| \leq c h_j^m \begin{cases} 1 & \text{if } m < 1-\nu, \\ 1 + |\log t_j| & \text{if } m = 1-\nu, \\ t_j^{1-\nu-m} & \text{if } m > 1-\nu, \quad j = 1, \dots, N. \end{cases}$$

Here c is a positive constant not depending on j and N, $h_j = t_j - t_{j-1}$ and t_j and $\mathcal{P}_N = \mathcal{P}_{N,m}$ are given by (3.1) and (3.7), respectively.

Lemma 4.2. *Let $\gamma < 0$ and β be real numbers, and let $N \geq 2$ be an integer. Then for all $l \in \mathbb{N}$ satisfying $2 \leq l \leq N$ the estimates*

$$\sum_{j=1}^{l-1} j^\beta (l-j)^\gamma \leq c \begin{cases} 1 & \text{if } \beta + \gamma < -1 \text{ and } \beta < 0, \\ N^\beta & \text{if } \beta \geq 0 \text{ and } \gamma < -1, \\ N^{\beta+\gamma+1} & \text{if } \beta + \gamma \geq -1 \text{ and } \gamma > -1, \end{cases}$$

$$\sum_{j=1}^{l-1} j^\beta (l-j)^\gamma \log\left(\frac{N}{j}\right) \leq c \begin{cases} \log N & \text{if } \beta + \gamma < -1 \text{ and } \beta \leq 0, \\ N^\beta & \text{if } \beta > 0 \text{ and } \gamma < -1, \\ N^{\beta+\gamma+1} & \text{if } \beta + \gamma > -1 \text{ and } \gamma > -1 \end{cases}$$

hold, where c is a positive constant which does not depend on l and N.

Lemma 4.3. *Let $z \in C^{q,\nu}(0,b]$, where $-\infty < \nu < 1$ and $q = m + \min\{m, \lceil\alpha\rceil\}$, with some $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}, \alpha > 0$. Moreover, assume that a quadrature approximation*

$$\int_0^1 g(x)\mathrm{d}x \approx \sum_{k=1}^{m} w_k g(\eta_k) \tag{4.3}$$

with the knots $\{\eta_k\}$ satisfying (3.3) and appropriate weights $\{w_k\}$ is exact for all polynomials of degree $q-1$. Then, for all values of the grid parameter $r \in [1,\infty)$ in (3.1), we have

$$\|J^\alpha(z - \mathcal{P}_N z)\|_\infty \le c \begin{cases} E_N(m,\alpha,\nu,r) & \text{if } 0 < \alpha < 1, \\ \Theta_N(m+\alpha,\nu,r) & \text{if } 1 \le \alpha \le m, \\ \Theta_N(2m,\nu,r) & \text{if } \alpha \ge m. \end{cases} \tag{4.4}$$

Here $\mathcal{P}_N$ is given by (3.7) and

$$E_N(m,\alpha,\nu,r) = \begin{cases} N^{-r(1+\alpha-\nu)} & \text{for } 1 \le r < \dfrac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}(1+\log N) & \text{for } r = \dfrac{m+\alpha}{1+\alpha-\nu} = 1, \\ N^{-m-\alpha} & \text{for } r = \dfrac{m+\alpha}{1+\alpha-\nu} > 1 \\ & \text{or } r > \dfrac{m+\alpha}{1+\alpha-\nu}, \end{cases} \tag{4.5}$$

$$\Theta_N(q,\nu,r) = \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \le r < \dfrac{q}{2-\nu}, \\ N^{-q}(1+\log N) & \text{for } r = \dfrac{q}{2-\nu} \ge 1, \\ N^{-q} & \text{for } r > \dfrac{q}{2-\nu}. \end{cases} \tag{4.6}$$

Proof. Fix $t \in (0,b]$. Let $l \in \{0,1,\ldots,N-1\}$ be such that $t \in (t_l, t_{l+1}]$. Then

$$J^\alpha(z - \mathcal{P}_N z)(t) = \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{l+1} F_j(t) \tag{4.7}$$

where

$$F_j(t) = \int_{t_{j-1}}^{t_j} (t-s)^{\alpha-1}(z - \mathcal{P}_N z)(s)\mathrm{d}s, \quad j = 1,\ldots,l, \tag{4.8}$$

$$F_{l+1}(t) = \int_{t_l}^{t} (t-s)^{\alpha-1}(z - \mathcal{P}_N z)(s)\mathrm{d}s.$$

Since $z \in C^{q,\nu}(0,b] \subset C^{m,\nu}(0,b]$, it follows from Lemma 4.1 that

$$|F_{l+1}(t)| \le h_{l+1}^\alpha \sup_{t_l < s < t_{l+1}} |(z - \mathcal{P}_N z)(s)| \le c h_{l+1}^{m+\alpha} \begin{cases} 1 & \text{if } m < 1-\nu, \\ 1 + |\log t_{l+1}| & \text{if } m = 1-\nu, \\ t_{l+1}^{1-\nu-m} & \text{if } m > 1-\nu. \end{cases}$$

Throughout the proof of Lemma 4.3 we assume that the generic constants c and c_1 are independent of $N \in \mathbb{N}$, $j \in \{1,\ldots,N\}$ and $t \in (0,b]$ and consequently also of l. As (see (3.1))

$$t_j = bj^r N^{-r}, \quad 0 < h_j = t_j - t_{j-1} \le brj^{r-1}N^{-r} \le brN^{-1}, \; j = 1,\ldots,N, \tag{4.9}$$

then

$$|F_{l+1}(t)| \le cN^{-(m+\alpha)} \begin{cases} 1 & \text{for } m < 1-\nu, \\ 1 + \log N & \text{for } m = 1-\nu. \end{cases}$$

Further, for $m > 1-\nu$ we obtain

$$\begin{aligned} |F_{l+1}(t)| &\le cN^{-r(1+\alpha-\nu)}(l+1)^{r(1+\alpha-\nu)-(m+\alpha)} \\ &\le c \begin{cases} N^{-r(1+\alpha-\nu)} & \text{if } 1 \le r \le \dfrac{m+\alpha}{1+\alpha-\nu}, \\ N^{-(m+\alpha)} & \text{if } r > \dfrac{m+\alpha}{1+\alpha-\nu}. \end{cases} \end{aligned}$$

Summarizing all these cases we get

$$|F_{l+1}(t)| \le cE_N(m, \alpha, \nu, r). \tag{4.10}$$

Since for $l \ge 1$ we have $t - t_{l-1} \le h_l + h_{l+1} \le h_l \max\{2^r, 3^{r-1} + 1\}$, in a similar way we obtain also the estimate

$$|F_l(t)| \le cE_N(m, \alpha, \nu, r). \tag{4.11}$$

If $l \ge 2$ then it is necessary to estimate yet $F_j(t),\ j = 1, \dots, l-1$. For $\phi(s) = (t-s)^{\alpha-1}$ we can use in $(t_{j-1}, t_j),\ j = 1, \dots, l-1$, the Taylor expansion

$$\phi(s) = \sum_{i=0}^{a-1} \frac{1}{i!}\phi^{(i)}(t_j)(s-t_j)^i + \frac{1}{a!}\phi^{(a)}(\sigma)(s-t_j)^a, \quad s \in (t_{j-1}, t_j),$$

where $a = \min\{m, \lceil\alpha\rceil\}$ and $\sigma = \sigma(s) \in (t_{j-1}, t_j)$. This together with (4.8) yields

$$\sum_{i=1}^{l-1} F_j(t) = \sum_{i=0}^{a} G_i(t) \tag{4.12}$$

where

$$G_i(t) = \sum_{j=1}^{l-1} \frac{1}{i!}\phi^{(i)}(t_j) \int_{t_{j-1}}^{t_j} (s-t_j)^i (z - \mathcal{P}_N z)(s)\mathrm{d}s, \quad i = 0, \dots, a-1,$$

$$G_a(t) = \sum_{j=1}^{l-1} \frac{1}{a!} \int_{t_{j-1}}^{t_j} \phi^{(a)}(\sigma)(s-t_j)^a (z - \mathcal{P}_N z)(s)\mathrm{d}s.$$

In order to derive the estimation (4.4) we introduce the interpolation operators $\mathcal{P}_{N,m+i}$ $(i = 1, \dots, a)$ as follows. We add to the parameters $\eta_1, \dots, \eta_m$ (see (4.3)) parameters $\eta_{m+1}, \dots, \eta_{m+i}$; the choice of the last ones in $[0, 1]$ is arbitrary, but we assume that they are somehow fixed and are different from $\eta_1, \dots, \eta_m$. In analogy to $\mathcal{P}_N = \mathcal{P}_{N,m}$ we determine $\mathcal{P}_{N,m+i},\ i = 1, \dots, a$, from the following conditions:

$$\mathcal{P}_{N,m+i}v \in S_{m+i-1}^{(-1)}(\Pi_N), \quad (\mathcal{P}_{N,m+i}v)(t_{jk}) = v(t_{jk}),\ k = 1, \dots, m+i,\ j = 1, \dots, N,$$

where $v \in C[0, b]$ and

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \dots, m+i,\ j = 1, \dots, N.$$

Since $q = m + a$ and the quadrature approximation (4.3) is exact for all polynomials of degree $q - 1$, we have

$$\int_{t_{j-1}}^{t_j} (s-t_j)^i (\mathcal{P}_N z)(s)\mathrm{d}s = \int_{t_{j-1}}^{t_j} (s-t_j)^i (\mathcal{P}_{N,q-i} z)(s)\mathrm{d}s, \quad i = 0, \dots, a-1,\ j = 1, \dots, l-1,$$

$$G_i(t) = \sum_{j=1}^{l-1} \frac{1}{i!}\phi^{(i)}(t_j) \int_{t_{j-1}}^{t_j} (s-t_j)^i (z - \mathcal{P}_{N,q-i} z)(s)\mathrm{d}s, \quad i = 0, \dots, a-1.$$

If $0 \le i \le \min\{a, \alpha\} - 1$ then

$$|\phi^{(i)}(t_j)| = |(\alpha-1)\dots(\alpha-i)(t-t_j)^{\alpha-i-1}| \le c, \quad j = 1, \dots, l-1,$$

and with help of Lemma 4.1 (where in the role $\mathcal{P}_N$ is $\mathcal{P}_{N,q-i}$) we get

$$|G_i(t)| \le c \sum_{j=1}^{l-1} h_j^{i+1} h_j^{q-i} \begin{cases} 1 & \text{if } q - i < 1 - \nu, \\ 1 + |\log t_j| & \text{if } q - i = 1 - \nu, \\ t_j^{1-\nu-q+i} & \text{if } q - i > 1 - \nu. \end{cases}$$

Using (4.9) we obtain for $0 \le i \le \min\{a, \alpha\} - 1$ that

$$|G_i(t)| \le c \begin{cases} N^{-r(2+i-\nu)} & \text{if } 1 \le r < \dfrac{q}{2+i-\nu}, \\ N^{-q}(1 + \log N) & \text{if } r = \dfrac{q}{2+i-\nu} \ge 1, \\ N^{-q} & \text{if } r > \dfrac{q}{2+i-\nu}, \end{cases}$$

from which it follows that

$$|G_i(t)| \le c\Theta_N(q, \nu, r), \quad 0 \le i \le \min\{a, \alpha\} - 1,\ l \ge 2. \tag{4.13}$$

If $\alpha = 1, \ldots, m$ then $a = \alpha,\ q = m + \alpha,\ E_N(m, \alpha, \nu, r) \le \Theta_N(m + \alpha, \nu, r)$, $G_a(t) = 0$ and from (4.7) and (4.10)–(4.13) it follows that

$$\|J^\alpha(z - \mathcal{P}_N z)\|_\infty \le c\Theta_N(m + \alpha, \nu, r), \quad \alpha = 1, \ldots, m,$$

i.e. the estimate (4.4) is true for $\alpha = 1, \ldots, m$. If $\alpha \ge m$ then

$$\|J^\alpha(z - \mathcal{P}_N z)\|_\infty = \|J^{\alpha-m} J^m(z - \mathcal{P}_N z)\|_\infty \le c\|J^m(z - \mathcal{P}_N z)\|_\infty \le c_1\Theta_N(2m, \nu, r),$$

i.e. the estimate (4.4) is valid also for $\alpha \ge m$.

Consider, finally, the case if $\alpha \in (0, m)$ and $\alpha \notin \mathbb{N}$. Since in this case $\alpha < a$ and $t - t_j \ge t_l - t_j \ge (l - j)h_j$, $j = 1, \ldots, l - 1$, $l \ge 2$, with the aid of Lemma 4.1 we get that

$$\begin{aligned} |G_{a-1}(t)| &= \left| \sum_{j=1}^{l-1} \frac{1}{(a-1)!} \phi^{(a-1)}(t_j) \int_{t_{j-1}}^{t_j} (s - t_j)^{a-1}(z - \mathcal{P}_{N,m+1} z)(s)\mathrm{d}s \right| \\ &\le c \sum_{j=1}^{l-1} (l-j)^{\alpha-a} h_j^{\alpha-a} h_j^a h_j^{m+1} \begin{cases} 1 & \text{if } m+1 < 1-\nu, \\ 1 + |\log t_j| & \text{if } m+1 = 1-\nu, \\ t_j^{1-\nu-m-1} & \text{if } m+1 > 1-\nu. \end{cases} \end{aligned}$$

This together with (4.9) and Lemma 4.2 yields

$$|G_{a-1}(t)| \le c \begin{cases} N^{-r(1+\alpha-\nu)} & \text{if } 1 \le r \le \dfrac{m+a}{1+\alpha-\nu}, \\ N^{-(m+a)} & \text{if } r \ge \dfrac{m+a}{1+\alpha-\nu}. \end{cases}$$

Therefore

$$|G_{a-1}(t)| \le cE_N(m, \alpha, \nu, r), \quad \alpha \in (0, m),\ \alpha \notin \mathbb{N},\ l \ge 2. \tag{4.14}$$

In a similar way we obtain

$$\begin{aligned} |G_a(t)| &\le c \sum_{j=1}^{l-1} (l-j)^{\alpha-a-1} h_j^{\alpha-a-1} h_j^{a+1} \sup_{t_{j-1}<s<t_j} |(z - \mathcal{P}_N z)(s)| \\ &\le c_1 E_N(m, \alpha, \nu, r), \quad \alpha \in (0, m),\ \alpha \notin \mathbb{N},\ l \ge 2. \end{aligned} \tag{4.15}$$

From (4.7), (4.10)–(4.12), (4.14) and (4.15) it follows that the estimate (4.4) holds for $\alpha \in (0, 1)$. If $\alpha \in (1, m)$ then $E_N(m, \alpha, \nu, r) \le \Theta_N(m+\alpha, \nu, r)$ and $\Theta_N(q, \nu, r) \le \Theta_N(m+\alpha, \nu, r)$. This together with (4.7) and (4.10)–(4.15) yields (4.4) also in the case $\alpha \in (1, m)$. □

Remark 4.1. If $z \in C^{m,\nu}(0, b]$ then the estimation (4.4) is valid also in the case $\alpha = 0$, i.e. $\|z - \mathcal{P}_N z\|_\infty \le E_N(m, 0, \nu, r)$ (see [2]).

Now we are ready to study the superconvergence properties of our method.

Theorem 4.2. *Let the following conditions be fulfilled:*

(i) $\mathcal{P}_N = \mathcal{P}_{N,m}$ $(N, m \in \mathbb{N})$ *is defined by* (3.7) *where the interpolation nodes* (3.2) *with grid points* (3.1) *and parameters* (3.3) *are used;*
(ii) *Eq.* (1.1) *satisfies the assumptions of Theorem* 2.1 *with* $q = m + \min\{m, \lceil \alpha_p - \alpha_{p-1} \rceil\}$;
(iii) *the quadrature approximation* (4.3) *is exact for all polynomials of degree* $q - 1$.

Then problem (1.1), (1.2) *has a unique solution* $y \in C^{n-1}[0, b]$ *with* $D_*^{\alpha_p} y \in C^{q,\nu}(0, b]$ *and there exists an integer* N_0 *such that, for* $N \ge N_0$, *Eq.* (3.6) *possesses a unique solution* $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ *and the following error estimates hold:*

(1) *if* $0 \le i \le \alpha_{p-1}$ *then*

$$\|y^{(i)} - y_N^{(i)}\|_\infty \le c \begin{cases} E_N(m, \alpha_p - \alpha_{p-1}, \nu, r) & \text{for } 0 < \alpha_p - \alpha_{p-1} < 1, \\ \Theta_N(m + \alpha_p - \alpha_{p-1}, \nu, r) & \text{for } 1 \le \alpha_p - \alpha_{p-1} \le m, \\ \Theta_N(2m, \nu, r) & \text{for } \alpha_p - \alpha_{p-1} \ge m; \end{cases} \tag{4.16}$$

(2) *if* $\alpha_{p-1} \le i \le n - 1$ *then*

$$\|y^{(i)} - y_N^{(i)}\|_\infty \le c \begin{cases} E_N(m, \alpha_p - i, \nu, r) & \text{for } 0 < \alpha_p - i < 1, \\ \Theta_N(m + \alpha_p - i, \nu, r) & \text{for } 1 \le \alpha_p - i \le m, \\ \Theta_N(2m, \nu, r) & \text{for } \alpha_p - i \ge m. \end{cases} \tag{4.17}$$

Here $r \in [1, \infty)$ *is the grading exponent of the grid (see* (3.1)*) and* $\nu,\ y_N,\ E_N$ *and* Θ_N *are defined by* (2.6), (3.4), (4.5) *and* (4.6), *respectively.*

Table 5.1
Results in the case $m = 2$, $\eta_1 = (3 - \sqrt{3})/6$, $\eta_2 = 1 - \eta_1$.

N	$r = 1$		$r = 2$		$r = 2.5$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	8.5E−4	3.46	1.8E−4	9.37	3.6E−4	7.24
8	2.3E−4	3.63	1.8E−5	9.90	4.0E−5	8.98
16	6.4E−5	3.65	1.9E−6	9.80	4.7E−6	8.57
32	2.1E−5	3.03	2.3E−7	8.11	5.9E−7	7.86
64	7.7E−6	2.77	3.8E−8	6.08	8.4E−8	7.04
128	2.7E−6	2.78	6.8E−9	5.51	1.3E−8	6.42
N	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$	$\hat{\varepsilon}_N$	$\hat{\varrho}_N$
4	1.9E−2	1.84	5.4E−3	3.46	5.3E−3	4.68
8	1.0E−2	1.85	1.5E−3	3.61	9.6E−4	5.48
16	5.4E−3	1.87	4.0E−4	3.76	1.7E−4	5.57
32	2.8E−3	1.89	1.0E−4	3.87	3.2E−5	5.40
64	1.5E−3	1.91	2.6E−5	3.93	5.7E−6	5.54
128	7.7E−4	1.93	6.6E−6	3.96	1.0E−6	5.61

Proof. Due to Theorem 2.1 Eq. (2.4) has a unique solution $z = D_*^{\alpha_p} y \in C^{q,\nu}(0, b] \subset C[0, b]$ and Eq. (1.1) with (1.2) possesses a unique solution $y = J^{\alpha_p} z + Q \in C^{n-1}[0, b]$. It follows from Theorem 4.1 that for sufficiently large N Eq. (3.6) has a unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$. Let $\hat{z}_N = Tz_N + \tilde{f}$ where $\tilde{f}$ and T are defined by (2.2) and (2.5), respectively. Then $\mathcal{P}_N \hat{z}_N = z_N$ and in a similar way as in the proof of Theorem 4.2 of [15] we can show that there exists an integer N_0 such that, for every $N \geq N_0$,

$$\|z - \hat{z}_N\|_\infty \leq c\|T(z - \mathcal{P}_N z)\|_\infty \leq c_1 \sum_{i=0}^{p-1} \|J^{\alpha_p - \alpha_i}(z - \mathcal{P}_N z)\|_\infty.$$

Since

$$\|J^{\alpha_p - \alpha_i}(z - \mathcal{P}_N z)\|_\infty \leq c\|J^{\alpha_p - \alpha_{p-1}}(z - \mathcal{P}_N z)\|_\infty, \quad i = 0, \ldots, p-1,$$

we obtain with the help of Lemma 4.3 that, for $N \geq N_0$,

$$\|z - \hat{z}_N\|_\infty \leq c \begin{cases} E_N(m, \alpha_p - \alpha_{p-1}, \nu, r) & \text{if } 0 < \alpha_p - \alpha_{p-1} < 1, \\ \Theta_N(m + \alpha_p - \alpha_{p-1}, \nu, r) & \text{if } 1 \leq \alpha_p - \alpha_{p-1} \leq m, \\ \Theta_N(2m, \nu, r) & \text{if } \alpha_p - \alpha_{p-1} \geq m. \end{cases} \tag{4.18}$$

Further, we have $z_N = \mathcal{P}_N \hat{z}_N$, $z - z_N = (z - \mathcal{P}_N z) + \mathcal{P}_N(z - \hat{z}_N)$ and

$$\|y^{(i)} - y_N^{(i)}\|_\infty \leq \|J^{\alpha_p - i}(z - z_N)\|_\infty \leq \|J^{\alpha_p - i}(z - \mathcal{P}_N z)\|_\infty + c\|z - \hat{z}_N\|_\infty,$$

where $i = 0, \ldots, n-1$. This together with (4.4) and (4.18) yields the estimates (4.16) and (4.17). □

5. Numerical experiments

Let us consider the following Cauchy problem:

$$D_*^{1.5} y(t) + 2D_*^1 y(t) + 3\sqrt{t} D_*^{0.5} y(t) + (1 - t)y(t) = f(t), \quad y(0) = y'(0) = 0, \tag{5.1}$$

where

$$f(t) = \frac{2}{\Gamma(1.5)} t^{0.5} + 4t + \frac{4}{\Gamma(1.5)} t^2 + (1 - t)t^2.$$

It is easy to check that all assumptions of Theorem 2.1 are fulfilled for $b > 0$ with $\nu = 0.5$ and arbitrary $q \in \mathbb{N}$. The exact solution of (5.1) is $y(t) = t^2$ and thus $D_*^{1.5} y(t) = (2/\Gamma(1.5))t^{0.5}$.

Problem (5.1) was solved numerically by the collocation method (3.5) on the interval [0, 1] using $m = 2$ and $\eta_1 = (3 - \sqrt{3})/6$, $\eta_2 = 1 - \eta_1$, the knots of the Gaussian quadrature formula (4.3). In Table 5.1 some results for different values of the parameters N and r are presented. The errors ε_N and $\hat{\varepsilon}_N$ in Table 5.1 are calculated as follows:

$$\varepsilon_N = \max_{j=1,\ldots,N} \max_{k=0,\ldots,10} |y(\tau_{jk}) - y_N(\tau_{jk})|,$$

$$\hat{\varepsilon}_N = \max_{j=1,\ldots,N} \max_{k=1,2} |z(t_{jk}) - z_N(t_{jk})|.$$

Here, $y_N = J^{1.5} z_N$, $z = D_*^{1.5} y$, $\tau_{jk} = t_{j-1} + k(t_j - t_{j-1})/10$, $k = 0, \ldots, 10$, $j = 1, \ldots, N$, and grid points t_j and collocation points t_{jk} are determined by (3.1) and (3.2), respectively. The ratios $\varrho_N = \varepsilon_{N/2}/\varepsilon_N$ and $\hat{\varrho}_N = \hat{\varepsilon}_{N/2}/\hat{\varepsilon}_N$, characterizing the observed convergence rate, are also presented.

Since $z_N(t_{jk}) = \hat{z}_N(t_{jk}),\ k = 1, 2,\ j = 1, \ldots, N$, we obtain from (4.16) and (4.18) that for sufficiently large N

$$\max\{\varepsilon_N, \hat{\varepsilon}_N\} \le c \begin{cases} N^{-r} & \text{if } 1 \le r < 2.5, \\ N^{-2.5} & \text{if } r \ge 2.5. \end{cases}$$

Therefore the values of the ratios ϱ_N and $\hat{\varrho}_N$ for $r = 1,\ r = 2$ and $r = 2.5$ ought to be approximately 2, 4 and $2^{2,5} \approx 5, 66$, respectively. As we can see from Table 5.1 the estimate (4.18) expresses well enough the actual rate of convergence of $\hat{z}_N$ to z but the convergence of $y_N = J^{1.5} z_N$ to y is somewhat faster than predicted by the estimate (4.16).

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